
TOPOLOGICAL SECTORS OF SPIN 1 THEORIES IN 2+1 DIMENSIONS

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*Dedicated to the memory of Professor Carlos Aragone
founder of the Group of Relativity and Fields at USB*

ABSTRACT

It is shown that the Topological Massive and “Self-dual” theories, which are known to provide locally equivalent descriptions of spin 1 theories in 2+1 dimensions, have different global properties when formulated over topologically non-trivial regions of space-time. The partition function of these theories, when constructed on an arbitrary Riemannian manifold, differ by a topological factor, which is equal to the partition function of the pure Chern-Simons theory. This factor is related to the space of solutions of the field equations of the Topological Massive Theory for which the connection is asymptotically flat but not gauge equivalent to zero. A new covariant, first order, gauge action, which generalize the “Self-dual” action, is then proposed. It is obtained by sewing local self-dual theories. Its global equivalence to the Topological Massive gauge theory is shown.

Vector and tensor gauge theories [1-5] in three dimensional space time enjoy very special properties arising from gauge invariant, topologically non trivial terms which provide masses for the gauge fields. These topological terms are related to the Chern-Simons characteristic classes, which may be obtained from four dimensional Pontryagin invariants and also arise naturally from the four dimensional topological BF theory [6,7]. These three topological functionals: Chern-Simons, Pontryagin and BF actions are just the starting point for most of the Topological Quantum Field Theories [6,8-11]. We are going to discuss one of these properties, enjoyed by the three dimensional vector gauge field theories, related to the topological terms mentioned above.

It is known that the spin 1 theory in 2+1 dimensions may be described by two covariant actions: The Topological Massive (TM) gauge action[2] and the, first order, Self Dual (SD) action[12,13]. Also, it has been shown that the SD action corresponds to a gauge fixed version of the (TM) theory [14-15]. This gives rise to the possibility of constructing two different covariant effective actions from the same gauge theory. One of these covariant gauge fixed actions is of second order in derivatives and arises by imposing the Lorentz covariant gauge fixing condition which involves, as usual, a Lagrange multiplier of the canonical formulation. The other one, the SD formulation, results from a canonical gauge fixing procedure which is not Lorentz covariant, however the full action can be rewritten in a Lorentz covariant form[15]. According to the BFV arguments, the partition functions and, in general, the physical observables are independent of the gauge fixing conditions within the admissible set. Under assumption of simply connectedness of the base manifold everything is equivalent [14-16], however once we consider both effective actions over a topologically non trivial base manifold, where the topological Chern-Simons structure may contribute to the observables of the theory, the relation between both formulations is not of a trivial equivalence. In this case, only the gauge fixed action with the Lorentz term preserves the global properties of the original gauge theory. However, both theories describe the same propagating physical degrees of freedom and have the same local properties. This is a novel feature not enjoyed by any known field theory in four dimensions.

We are going to explicitly show this global difference by comparison of the partition functions associated to both actions when formulated over a Riemannian base manifold and show that they differ by a topological factor equal to the partition function of the pure Chern-Simons action, which, as is well known, may be expressed in terms of the topological Ray-Singer torsion. This topological factor has its origin in the difference

between the space of solutions of the field equations associated to both theories. In fact, the space of flat connections is a solution of the field equations for the TM gauge theory, while the only flat connection which is a solution of the SD theory is the trivial one. For simply connected regions of space-time both spaces of solutions are identical but for topologically non trivial base manifolds the space of flat connections admits non-trivial configurations. Gauge inequivalent flat connections are characterized by their holonomy around non-contractible loops. This is equivalent to specifying homomorphisms from the fundamental Π_1 group into the structure group ($U(1)$ in our case). The topological index

$$I = \oint a, \quad (1)$$

evaluated for asymptotically flat solutions, is zero in the case of the SD theory and is $\neq 0$ for the non trivial solutions of the TM gauge theory (explicit solutions have been found in [4,5,16]). This space of asymptotically flat solutions correspond exactly to the Chern-Simons (CS) classical solutions, which are connected with the description of anyons [17,18] (see the reviews in [17] and the references there in), and because of this reason the solutions with non trivial topological index I are said to have “anyonic behaviour” .

After showing this global difference in the two formulations, we are going to present a new covariant, first order, gauge action which generalizes the SD one and is locally and globally equivalent to the TM action.

Let us start our discussion by showing briefly the canonical equivalence between both theories, over a simply connected region of space-time. The Lagrangian density of the SD theory is

$$L = \frac{m^2}{2} a_\mu a^\mu - \frac{m}{2} \varepsilon^{\mu\nu\rho} a_\mu \partial_\nu a_\rho, \quad (2)$$

and the canonical Hamiltonian density associated to it is given by

$$\mathcal{H}_0 = \frac{m^2}{2} a_i a_i + \frac{1}{2} (\varepsilon_{ij} \partial_i a_j)^2 \quad (3)$$

subject to two second class constraints

$$\theta_i = P_i - \frac{m}{2} \varepsilon_{ik} a_k = 0 \quad (4)$$

where P^i is the conjugate momenta associated to a_i (we are using metric signature $(+--)$). It was noticed, in [15], that they may be interpreted as a first class constraint θ and its

associated gauge fixing condition χ ,

$$\theta = -\partial_i \theta_i = -\partial_i P_i + \frac{m}{2} \varepsilon_{ij} \partial_i a_j \quad (5)$$

$$\chi = \varepsilon^{ij} \partial_i \theta_j = \varepsilon_{ij} \partial_i P_j + \frac{m}{2} \partial_k a_k. \quad (6)$$

The system (2) may, then, be considered as a gauge theory governed by the Hamiltonian density

$$\widehat{\mathcal{H}}_0 = \frac{1}{2} P_k P_k + \frac{m}{2} \varepsilon_{ij} P_i a_j + \frac{m^2}{8} a_k a_k + \frac{1}{2} (\varepsilon_{ij} \partial_i a_j)^2, \quad (7)$$

(which reduces to (3), under (5) and (6)) subject to the first class constraint θ .

The quantized formulation of this new gauge system is equivalent to that of the original one [15]. The new Hamiltonian density (7) and the first class constraint θ , are just the ones which emerge from the canonical analysis of the TM gauge theory. Moreover, with χ as a gauge fixing condition the effective action is just the SD action. It is important to notice that (5) and (6) are equivalent to (4) provided the region of space time we are considering is simply connected. So both theories, the SD and the TM , are completely equivalent at the classical and quantum level on a simply connected region of space-time.

We now compare the partition function of the two theories. The global difference in the space of classical solutions is going to be reflected in the evaluation of the partition functions where a topological factor will arise, in the TM case, which is not present in the SD one. In order to detect this topological factor we consider the formulation of both theories in a general Riemannian 3-manifold background, M . We start considering the canonical formulation with the correct quantum measure and evaluate the partition function after integration of the conjugate momenta. At the end of the paper we will discuss briefly the zero modes contributions.

We have that for the SD theory

$$S^{SD} = \int_M d^3x \sqrt{g} a_\mu T^{\mu\nu} a_\nu \quad (8)$$

with

$$T^{\mu\nu} = \frac{m}{2} (mg^{\mu\nu} - \frac{1}{\sqrt{g}} \varepsilon^{\mu\lambda\nu} \mathcal{D}_\lambda) \quad (9)$$

where \mathcal{D}_λ is the covariant derivative, on the manifold. The partition function is then

$$Z_{SD} = \rho (\det \sqrt{g} T)^{-1/2} \quad (10)$$

where $\rho = (\det(m^2 \delta))^{1/2}$ arises from the Senjanovic-Fradkin measure term $\det^{1/2} \{\theta_i, \theta_j\}$.

For the TM theory, in the Lorentz gauge, the $BRST$ invariant effective action, on M , takes the form

$$S_{eff}^{TM} = \int_M d^3x \sqrt{g} [a_\mu S^{\mu\nu} a_\nu - mB\mathcal{D}_\mu a^\mu - m\bar{C}\mathcal{D}_\mu \partial^\mu C] \quad (11)$$

where

$$S^{\mu\nu} = \frac{1}{2} [(g^{\mu\nu} \mathcal{D}_\lambda \mathcal{D}^\lambda - \mathcal{D}^\nu \mathcal{D}^\mu) + \frac{m}{\sqrt{g}} \varepsilon^{\mu\lambda\nu} \mathcal{D}_\lambda] \quad (12)$$

and B, C, \bar{C} are, respectively the Lagrange multiplier, ghost and antighost fields introduced when the canonical $BRST$ procedure is applied. The degeneracy of $S^{\mu\nu}$ is easily observed because $S^{\mu\nu} \partial_\nu \lambda = 0$ for any 0-form λ . If we define the differential operator

$$C^{\mu\nu} = \frac{1}{m} \varepsilon^{\mu\nu\lambda} \partial_\nu \quad (13)$$

which is proportional to the kinetic operator of the pure Chern-Simons theory, is immediate to see that acting on 1-forms

$$T^{\mu\nu} \frac{1}{\sqrt{g}} C_\nu^\lambda = \frac{1}{\sqrt{g}} C^{\mu\nu} T_\nu^\lambda = S^{\mu\lambda} \quad (14)$$

this fact is going to be crucial when we compare the two partition functions. Before considering that point, we must notice that in this Riemannian 3-manifold M , one can define the Hodge dual $*$ which maps p-forms on $(3-p)$ -forms and satisfies $** = \mathbf{1}$. The adjoint of the exterior derivate d is $\delta = (-1)^p * d *$ when acting on p-forms, and satisfies $\delta^2 = 0$. Finally the Laplacian on p-forms (the Laplace-Beltrami operator) is defined to be $\Delta \equiv \delta d + d\delta$, as usual. So over a 0-form λ

$$\Delta_0 \lambda = -\mathcal{D}_\mu \partial^\mu \lambda \quad (15)$$

and over a 1-form V_μ

$$\Delta_1 V_\mu = -\mathcal{D}_\nu \mathcal{D}^\nu V_\mu + R_{\mu\nu} V^\nu \quad (16)$$

where $R_{\mu\nu}$ is the Ricci tensor.

The action can, then, be rewritten as

$$S_{eff}^{TM} = \int_M d^3x \sqrt{g} [\Phi^t K_B \Phi + m\bar{C} \Delta_0 C] \quad (17)$$

where

$$K_B \equiv \begin{pmatrix} S^{\mu\lambda} & \frac{m}{2} \partial^\mu \\ -\frac{m}{2} \mathcal{D}^\lambda & 0 \end{pmatrix} \quad (18)$$

and $\Phi^t = (a_\lambda \quad B)$. Then, the partition function will be[11]

$$Z_{TM} = \det(\sqrt{g}K_B)^{-1/2} \det m\delta \det \Delta_0. \quad (19)$$

To evaluate the determinat of K_B we take the square the operator which is diagonal

$$K_B^2 = \begin{pmatrix} S^{\mu\alpha} S_\alpha^\lambda - \frac{m^2}{4} \partial^\mu \mathcal{D}^\lambda & 0 \\ 0 & -\frac{m^2}{4} \mathcal{D}_\alpha \partial^\alpha \end{pmatrix} \quad (20)$$

For the part which act on 1-forms, it can be seen, using (14), that

$$S^{\mu\alpha} S_\alpha^\lambda - \frac{m^2}{4} \partial^\mu \mathcal{D}^\lambda = \frac{1}{m^2} T^{\mu\theta} (-g_{\theta\rho} \mathcal{D}_\alpha \mathcal{D}^\alpha + \mathcal{D}_\rho \mathcal{D}_\theta - \mathcal{D}_\theta \mathcal{D}_\rho) T^{\rho\lambda}, \quad (21)$$

so

$$K_B^2 = \begin{pmatrix} \frac{1}{m^2} T \Delta_1 T & 0 \\ 0 & \frac{m^2}{4} \Delta_0 \end{pmatrix}$$

and

$$Z_{TM} = \rho (\det \sqrt{g} T)^{-1/2} (\det \Delta_1)^{-1/4} (\det \Delta_0)^{3/4}. \quad (22)$$

Here ρ is the same factor as in the *SD* theory. The two partition functions differ, then, in a factor which is just the partition function of the pure Chern-Simons theory. This factor is related to the Ray-Singer torsion, $T(M_3)$, by $Z_{CS} = T(M_3)^{-1/2}$, and is metric independent [19,8]. For an even dimensional oriented compact manifold, without boundary, $T(M_{2m}) = 1$. This fact is obtained, from the scaling invariance in path integrals of some *BF* systems [8]. In odd dimensions, in distinction, those invariances, together with the Hodge duality property $(*\Delta = \Delta*)$, lead to identities that give no information about $T(M_{2m+1})$ [8]. In other direction, the two point function of Topological Field theories, whose partition function is the Ray-Singer torsion, can be used as a definition of generalized linking number between surfaces [7], which is connected with the concepts of fractional statistics [17,18,20].

The common factor between Z_{SD} and Z_{TM} could be expected because the self-dual equations of motion constitutes a minimal realization of the “Pauli-Lubanski”, and mass shell conditions for the spin 1 representations of the Poincaré group in 2+1 dimensions [21]. The extra factor is connected with the “topological” properties of the *TM* theory, and explains why, for the *SD* theory, there is no “anyonic behaviour”. This topological factor reduces to one, for a simply connected region of space time, where both theories are equivalent. In fact, assuming $M_3 = R \times M_2$ and M_2 simply connected we have

$$\det \Delta_1(M_3) = \det \Delta_1(M_2) \det \Delta_0. \quad (23)$$

Now, using Proposition 4 in [8], it can be seen that $\det \Delta_1(M_2) = (\det \Delta_0)^2$; hence

$$\det \Delta_1(M_3) = (\det \Delta_0)^3, \quad (24)$$

and

$$(\det \Delta_1)^{-1/4}(\det \Delta_0)^{3/4} = 1. \quad (25)$$

(10) and (22), show that the *TM* theory, which is locally equivalent to the *SD* theory, is globally different to it.

When the base manifold is RxM_2 , the occurrence of M_2 as a multiply connected manifold arises in various interesting models. The simplest one is when we couple minimally the *TM* or the *SD* theories to a source that consists of a charge particle at the origin. If the source has “dipole strength” σ , the static solutions, outside sources, for the *SD*, *TM* and pure *CS* theories, are related by [4,16]

$$a_0^{CS} = 0, \quad (26,a)$$

$$a_0^{SD} = a_0^{TM} = -(q + m\sigma)Y(mr), \quad (26,b)$$

$$\begin{aligned} a_i^{TM} &= a_i^{SD} + a_i^{CS} \\ &= \frac{q + m\sigma}{m} \epsilon_{ij} \partial_j Y(mr) + \left(-\frac{q}{m} \epsilon_{ij} \partial_j C(mr) + \partial_j \lambda\right), \end{aligned} \quad (26,c)$$

where the longitudinal part of a_i remains unfixed, in the *TM* and pure *CS* theories, because of the gauge freedom. $Y(mr)$ and the $C(mr)$ are, respectively, the Yukawa and Coulomb Green functions, i.e. $(-\Delta + m^2)Y(mr) = (-\Delta)C(mr) = \delta^2(r)$. Asymptotically $a_\mu^{TM} \sim a_\mu^{CS}$ and $a_\mu^{SD} \sim 0$. In the special case that $q + m\sigma = 0$, these last relations hold everywhere [4]. For both cases $F_{\mu\nu} = 0$ but only for the *TM* and *CS* theories the potential a_μ is closed and not exact. More precisely, the index I , (1), (for loops around the origin) becomes q/m as in the pure *CS* theory. This result is used to implement fractional statistics dynamically [14, 16-18].

The local relation with the *SD* solutions arises because a_i^{TM} can be rewritten as ($q + m\sigma = 0$)

$$a_i^{TM} = a_i^{CS} = \partial_i \left(\lambda - \frac{q\Theta}{2\pi m} \right), \quad (27)$$

where $\Theta = \text{arctg}(x_2/x_1)$ is a multivalued function. However, $\partial_i \Theta$ is a well defined 1-form on the punctured plane known as an Oersted-Amper 1-form [22]. This 1-form is closed

$(F_{ij} = 0)$, but not exact ($\oint \partial_i \Theta dx^i \neq 0$, for loops around the origin). The possibility of fixing gauge in such a way that $a_i = 0$ can only be performed on simply connected regions, but not globally (this gauge is commonly known as the singular gauge that eliminates the potential). So the non equivalence between both theories is also reflected in this special example. It can be shown that, by making a different kind of coupling, the self dual solutions can reproduce the TM ones, but it must be a non-local type of coupling [14,16]. If we want to obtain the topological sector of the space of solutions not present in the SD model it seems that one should consider patching and sewing “ SD formulations” over simply connected sectors of the base manifold. In order to do so, we start considering the functional integral of the TM theory. Its functional measure is

$$\delta^2(\theta)\delta^2(\chi) \det\{\theta, \chi\}, \quad (28)$$

where θ and χ are given by (5) and (6) respectively. This may be rewritten as

$$\langle \prod_{i=1}^2 \delta^2(\theta_i + m\epsilon_{ij}\omega_j) \det^{1/2}\{\theta_i, \theta_j\} \mu \rangle_{H_1}, \quad (29)$$

where $\omega_i dx^i$ is a 2 dimensional closed 1-form satisfying $\partial_i \omega_i = 0$. This condition only fixes the exact forms corresponding to a given cohomology class. Integration is done on the space of cohomology classes, H_1 with measure $\mu = Z_{CS}(R \times M_2)$. θ_i is defined as in (4). (29) constitutes the generalization of the arguments, used from (2) through (7), to a topological non trivial space.

After integration on the conjugated momenta we arrive to the functional integral associated to the following action

$$S' = \int d^3x \left[\frac{m^2}{2} (a_\mu + \omega_\mu)(a^\mu + \omega^\mu) - \frac{m}{2} (a_\mu + \omega_\mu) \epsilon^{\mu\nu\lambda} \partial_\nu (a_\lambda + \omega_\lambda) \right], \quad (30)$$

where the closed form $\omega_\mu dx^\mu$ satisfies the gauge condition $\partial^\mu \omega_\mu = 0$. a_μ and ω_μ are independent fields. Functional integration on ω_μ is performed in H_1 with measure μ . The functional integral of the TM theory is then equivalent to the functional integral associated to (30) which is also a gauge invariant action. The condition imposed to ω_μ can always be selected on any cohomology class. In the previous argument we assumed $M_3 = M_2 \times R$, in order to perform the canonical analysis. In the particular case that M_3 is simply connected $\omega_\mu = 0$ the usual SD formulation is regained.

The constraints associated to (30) are

$$P_i - \frac{m}{2} \epsilon_{ij} (a_j + \omega_j) = 0, \quad (31)$$

and

$$\langle \pi^I - P^j \Lambda_j^I \rangle_{M_2} = 0, \quad (32)$$

here P^j has the same meaning as before, while $\pi^I(t)$ is the conjugate momenta associated to $\alpha^I(t)$, where $\Lambda_i^I(x)$ is a basis of closed 1-forms, so

$$\omega_i(t, x) = \alpha^I(t) \Lambda_i^I(x). \quad (33)$$

Lastly $\langle \quad \rangle_{M_2}$ denotes integration on M_2 . We may fix the gauge transformations generated by (32) taking

$$\alpha^I(t) = \text{constant}. \quad (34)$$

then the condition on ω_μ reduces to $\partial_i \omega_i = 0$, which was the restriction obtained in (29).

The classical field equations arising from (30) are

$$mA^\mu - \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda = -m\omega^\mu, \quad (35)$$

which may be rewritten as

$$\epsilon_{\rho\gamma\mu} \partial^\gamma [ma^\mu - \epsilon^{\mu\nu\lambda} \partial_\nu a_\lambda] = 0. \quad (36)$$

(37) being the classical equations of the TM theory. Variations with respect to the α^I do not give any new equation of motion. The partition functions associated to the TM and the modified SD theories (in (30)) are also equal as we have mentioned. This feature may be shown from the analysis of the functional measure as we did, or by direct evaluation as in (10) and (22) where the volume of the zero modes of the CS operator must now be included. Details of this analysis will be reported elsewhere.

We have shown that the TM and SD theories, which are locally equivalent, have different global properties. The difference arises, as is known, at the classical level where non-trivial flat connections are solutions of the TM theory, while the trivial flat connection is the only admissible solution for the SD theory. We observe , then, by explicit evaluation, that the partition functions differ by a topological factor, the CS partition function associated to that topological sector of the space of solutions. Finally, we have constructed

a covariant extension of the SD theory, also of first order in derivatives, which is exactly equivalent, locally and globally, to the TM theory.

We have consider only the spin 1 abelian theory. We expect analogous results for other spins in $3 - D$. The case of spin 2 linear theories is particulary interesting since there are three equivalent linear theories with the same local physics [2,23] but clearly with different global properties [4,5]. Also there is a kind of factorization analogous to (14) and a gauge fixing procedure [24], connecting one theory to the other. This will be reported elsewhere.

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